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An efficient and accurate iterative stress solution for an infinite elastic plate around two elliptic holes, subjected to uniform loads on the hole boundaries and at infinity

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ABSTRACT

Using the Schwarz's alternating method and the Muskhelishvili's complex variable function techniques, an efficient and accurate stress solution for an infinite elastic plate around two elliptic holes, subjected to uniform loads on the hole boundaries and at infinity, is presented in this paper. The present algorithm can be used to compute the stress concentration factors (*SCF*), i.e., the ratio of the maximum tangential hoop stress to the applied uniform load, on the boundaries of the two elliptical holes of different sizes and layouts under different loading conditions, as illustrated in two numerical cases.

1. Introduction

The existence of two elliptic holes in solids is often encountered in the design of machinery manufacture, pressure vessels, aircrafts, and so on. The designers are usually interested in the stress states around the elliptic holes, especially for a general case in which uniform loads are applied both on the hole boundaries and at infinity. Jones and Hozos (1971) conducted photoelastic experiments on large square plates containing two neighboring elliptic holes with uniaxial and biaxial in-plate tensions applied at the outer edges, which provided interesting experimental results. By means of the Muskhelishvili's techniques and the Schwarz's alternating method (Muskhelishvili, 1953; Sokolnikoff, 1962), Ukadgaonker and Patil (1993) analyzed the stress state of a plate containing two elliptic holes subjected to uniform pressures and tangential shears on the hole boundaries, however, with a second approximation achieved. without considering loads at the hole edges, Zhang and Lu (1998) and Zhang et al. (2001, 2003) conducted a series of studies on two circular or multiple elliptic holes under uniform loads at infinity, using the Muskhelishvili's techniques and the Schwarz's alternating method. Based on the approximation of the redundant surface force vector as a series of complex variable, they deduced a generalized and approximated form of the stress boundary condition for all single hole problems during the iterative calculations.

So far, no exact or highly accurate numerical solutions have been reported in the literatures for an infinite elastic plate around two elliptic holes under uniform loads both along the hole boundaries and at infinity. In this paper, an alternative algorithm is proposed to find an efficient and accurate stress solution for the double elliptic hole problem, by means of the Schwarz's alternating method and the Muskhelishvili's techniques.

2. Basic procedures and convergent criterion

This paper attempts to solve the stress state for an infinite elastic plane containing two elliptic holes. As shown in Fig. 1, the holes can be of any dimension and location. However, the axes of the elliptic holes are parallel or orthogonal, and they are not reducible to cracks. σ_x^∞, σ_y^∞ and τ_{xy}^∞ denote the uniform normal tensions (tension is considered a positive stress here) and tangential shears with the orientations consistent with the global coordinate system. N_1 and S_1 indicate the uniform normal tensions (tensile positive) and tangential shears (clockwise positive) along the boundary of the hole No. 1, and N_2 and S_2 for the hole No. 2.

For the stress boundary value problem shown in Fig. 1, the use of the Schwarz's alternating method is in fact a process of repeated elimination of redundant surface traction induced by the solution of the former single hole problem. Starting from the hole No. 1, stress solution for an infinite region only containing the hole No. 1 under σ_x^∞, σ_y^∞ and τ_{xy}^∞ applied at the infinity and N_1 and S_1 on

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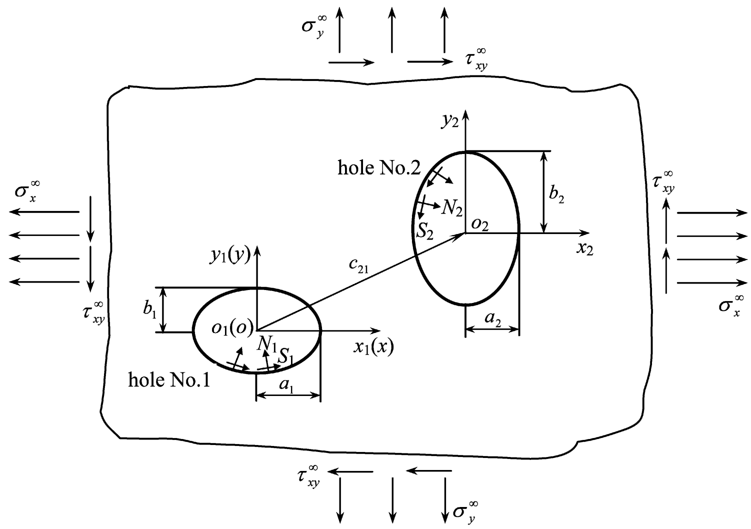

Fig. 1. An infinite elastic plate containing two elliptical holes subjected to any uniform stresses σ_x^∞, σ_y^∞ and τ_{xy}^∞ at the infinity, normal tractions N_1 and N_2 and tangential shears S_1 and S_2 on the boundaries.

the hole boundary, can be obtained by using the Muskhelishvili's complex variable function techniques. The solution satisfies the stress boundary conditions on the boundary of the hole No. 1 and at infinity, but causes redundant surface traction, in addition to the given boundary loads N_2 and S_2, on the boundary of the hole No. 2. The redundant surface traction can then be approximated by a complex series. To balance the redundant surface traction, a loading of opposite direction and equal magnitude is applied at the edge of the hole No. 2, which is called the reverse surface traction in the present paper. Then another single hole problem that an infinite plate only containing the hole No. 2 is loaded by the prescribed N_2 and S_2 and the reverse surface traction is solved. The stress solution at this stage satisfies the given stress boundary condition along the hole No. 2 and free-tractions at infinity, but creating a non-zero surface traction on the boundary of the hole No. 1. The surface traction is also redundant and should be again approximated by the complex series. Thus, one should solve the third single hole problem that the corresponding reverse surface traction in series form is applied at the edge of the hole No. 1. The redundant surface traction resulted on the boundary of the hole No. 2 can then be calculated. The process can be continued until the redundant surface traction on the two hole boundaries is small enough to satisfy the required stress accuracy.

From the above description, the solutions of every two single hole problems indicate two approximations (i.e., one cycle of iteration), and every single hole problem is solved in terms of the loading condition that only the reverse surface traction induced by the former single hole problem is applied along the hole boundary, except for the first iteration. The final stress solution is linear superposition of the stresses computed for all the single hole problems during the iterative calculations. The number of successive iterations, which is needed to achieve the required accuracy, depends upon sizes and configuration of two elliptic holes, and the loading conditions.

During iteration, the accuracy of stress results can be assessed by the extent that the redundant surface tractions approach to zeroes. With the increase in the number of iterations, the redundant surface traction is expected to be close to zero. Therefore, we use the convergent criterion below in the iterative calculations.

$$I = \frac{1}{NP}\sum_{k=1}^{NP}\sqrt{\bar{\sigma}_\rho^2 + \bar{\tau}_{\rho\theta}^2} \leqslant \Delta \tag{1}$$

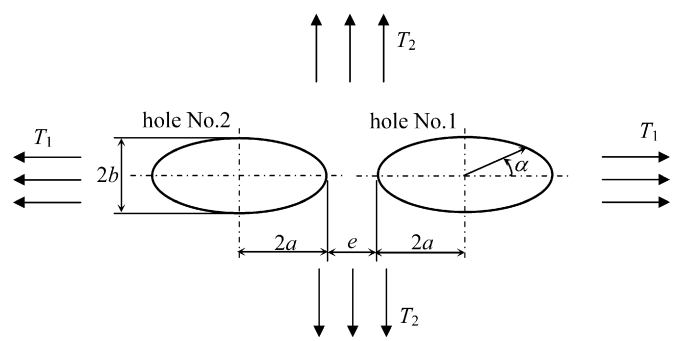

Fig. 2. Two identical elliptic holes in a flat plate loaded by two orthogonal uniform tensions T_1 and T_2 at infinity.

where $\bar{\sigma}_\rho$ and $\bar{\tau}_{\rho\theta}$ are the two components of the redundant surface traction; *NP* is the total evaluation point number at each hole edge and is set to be 360, which indicates a equal angle distribution of the evaluation points; Δ is a given value of the absolute accuracy. When the redundant surface tractions conform to (1), the stress result is considered to satisfy the prescribed stress boundary condition on the two holes' boundaries.

3. Comparison with photoelastic experimental results by Jones and Hozos (1971)

So far, no exact solution has been reported for a double elliptic hole problem in the literature. Jones and Hozos (1971) conducted a series of photoelastic experiments on large square plates containing two neighboring elliptic holes with uniaxial and biaxial in-plate tensions applied at the outer edges. The photoelastic sheets had the same size of 20 in × 20 in × 0.044 in, which were perforated by two identical elliptic holes and loaded by in-plane biaxial uniform tensions, vertically T_2 and horizontally T_1, around the outer edges of the sheets. The major and minor axes were 1.5 in ($2a$) and 0.75 in ($2b$), respectively, with the separation distances (e) of 0.128 in, 0.192 in, 0.357 in, 0.748 in, 1.494 in and 2.983 in, as indicated in Fig. 2. Different from plane problems of an infinite region involved in the present iterative solution, the boundaries of photoelastic sheets inevitably had some influences on the stress results around the two elliptic holes.

The stresses are calculated by the proposed iterative solution, with the same hole sizes, separation distances and in-plane load

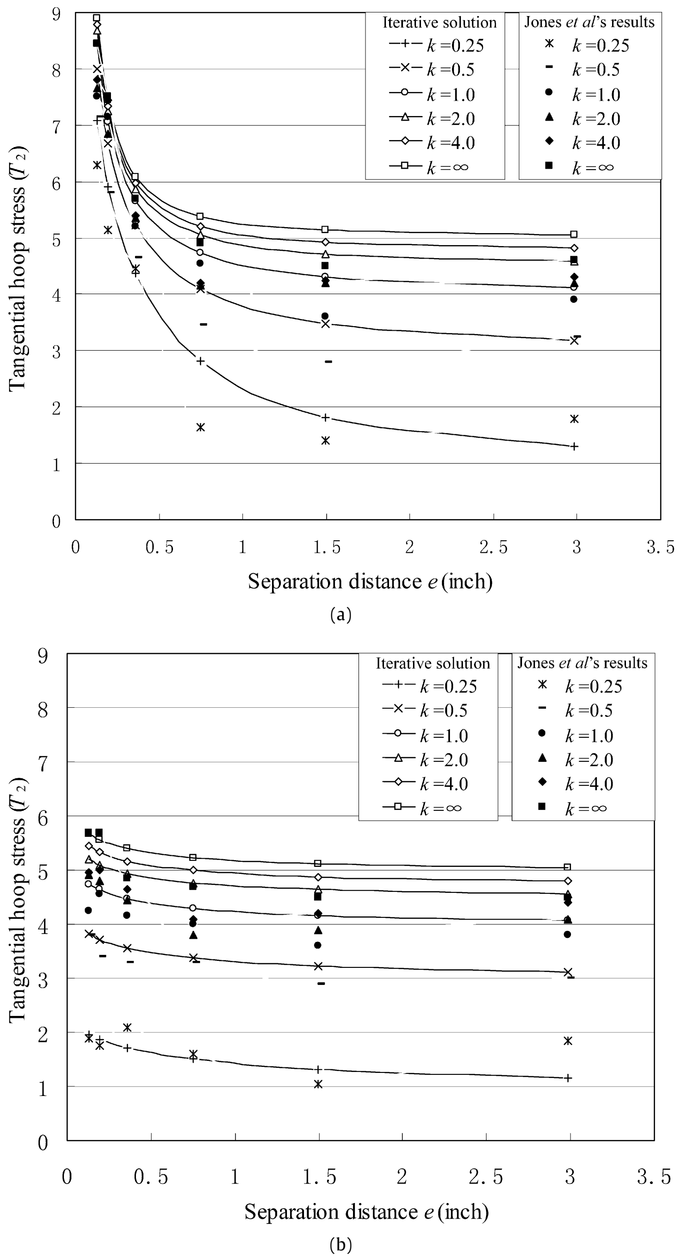

Fig. 3. Tangential hoop stresses σ_θ/T_2 at the ends of major and minor axes of hole No. 1, for various in-plane load ratio k ($k = T_2/T_1$). (a) At the left end of major axis of hole No. 1, i.e. $\alpha = \pi$. (b) At the right end of major axis of hole No. 1, i.e. $\alpha = 0$.

ratio k ($k = T_2/T_1$) as those in the photoelastic experiments. An accuracy of $\Delta = 1.0 \times 10^{-4} T_2$ is required. The requirement cannot be satisfied for two cases with smaller separation distances (i.e. $e = 0.128$ in and $e = 0.192$ in), but a relatively high accuracy between $3.3 \times 10^{-3} T_2$ and $6.9 \times 10^{-4} T_2$ can still be achieved. Due to the symmetry of the double hole problems, the authors only present the tangential hoop stresses σ_θ/T_2 at the ends of the major and minor axes (i.e., $\alpha = \pi$ and $\alpha = 0$) of the hole No. 1. As indicated in Fig. 3, the iterative stress solution reasonably agrees with the photoelastic experimental results, and most experimental results are smaller, especially for the stress concentration at the left end of major axis of hole No. 1 (see Fig. 3(a)).

4. Two numerical examples

Zhang et al. (2003) have provided iterative stress solution for an elastic plate around multiple elliptic holes with uniform tractions

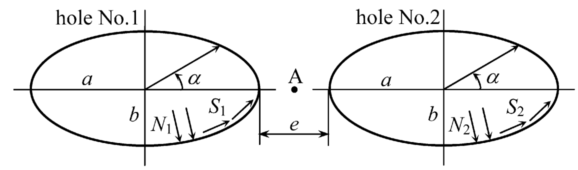

Fig. 4. Two identical elliptic holes ($a/b = 2$) in an infinite elastic plate, subjected to uniform normal pressures N_1, N_2 (negative) and uniform tangential shears S_1, S_2 (positive) on the hole boundaries.

at infinity, so the authors will only present two numerical examples with uniform loads on the boundaries of two elliptic holes in this paper. An accuracy of $\Delta = 1.0 \times 10^{-4} T$ (T is uniform tension at infinity) is utilized in the iterative solution for the two numerical examples.

4.1. *An elastic plate containing two identical elliptic holes under uniform normal pressures and tangential shears on the hole boundaries*

Now let us consider a problem for two identical elliptic holes in a plate subjected to uniform normal pressures (tension positive) and tangential shears (clockwise positive) on the hole boundaries (see Fig. 4). Three loading cases are illustrated: (1) $N_1 = N_2 = -1$ and $S_1 = S_2 = 0$, (2) $N_1 = N_2 = 0$ and $S_1 = S_2 = 1$, and (3) $-N_1 = S_2 = 1$ and $N_2 = S_1 = 0$. The stress concentration factors for the three loading cases are defined as $SCF = \sigma_{\theta\,\max}/|N_1|$, $SCF = \sigma_{\theta\,\max}/S_1$ and $SCF = \sigma_{\theta\,\max}/S_2$, respectively. Similar to the definitions of *SCF*, two biggest stress concentrations on the boundary of hole No. 1 are denoted by SCF1 and SCF2 for every loading case, respectively, and two biggest stress concentrations on the boundary of hole No. 2 are denoted by SCF3 and SCF4 for every loading case, respectively. The positions of the four stress concentrations may vary due to the differences of geometric or loading conditions, with their position numbered in Fig. 5 (a), (b) and (c).

The case (1) is a symmetrical double hole problem. As shown in Fig. 5(a), SCF1 (equal to SCF4) is larger than SCF2 (equal to SCF3). As the distance e/a increases, SCF1 and SCF2 gradually decrease to 3.0 and their positions are kept at 0 and 180 degrees on the boundary of the hole No. 1, respectively.

For the case (2), the stress field should be symmetrical about the geometrical center A, as shown in Fig. 4. SCF1 and SCF4 are equal and their positions are symmetrical about the point A, so are SCF2 and SCF3. As the distance e/a increases, the position of SCF1 varies from 164.69 to 165.94 degrees and SCF2 from 352.79 to 346.00 degrees.

Fig. 5(c) provides the curves of stress concentration factors versus the separation distance e/a, at the points of from No. 1 to No. 4 for the loading case (3), in which SCF2 is the largest one. As the distance e/a increases, SCF1 and SCF2 gradually decreases to 3.0, SCF3 and SCF4 to 1.5. As the distance e/a varies from 0.25 to 10, the four stress concentrations, SCF1 and SCF2, SCF3 and SCF4, have various positions of from 177.95 to 179.90 degrees, from 353.62 to 359.85 degrees along the hole No. 1, from 175.22 to 166.06 degrees and from 347.79 to 346.06 degrees along the hole No. 2, respectively.

4.2. *An elastic plate containing an elliptic hole and a circular hole with uniform loads on hole boundaries*

As plotted in Fig. 6, the major axis of an elliptic hole is 1 times longer than its minor axis (i.e., $a/b = 2.0$), and a circular hole of radius r is placed at a horizontal center-to-center distance of d and a vertical center-to-center distance of $d/3$ from the elliptic hole. Two loading cases are considered in this example, i.e., (1) $N_1 = N_2 = -1$ and $S_1 = S_2 = 0$; (2) $N_1 = N_2 = 0$ and $S_1 = S_2 = 1$.

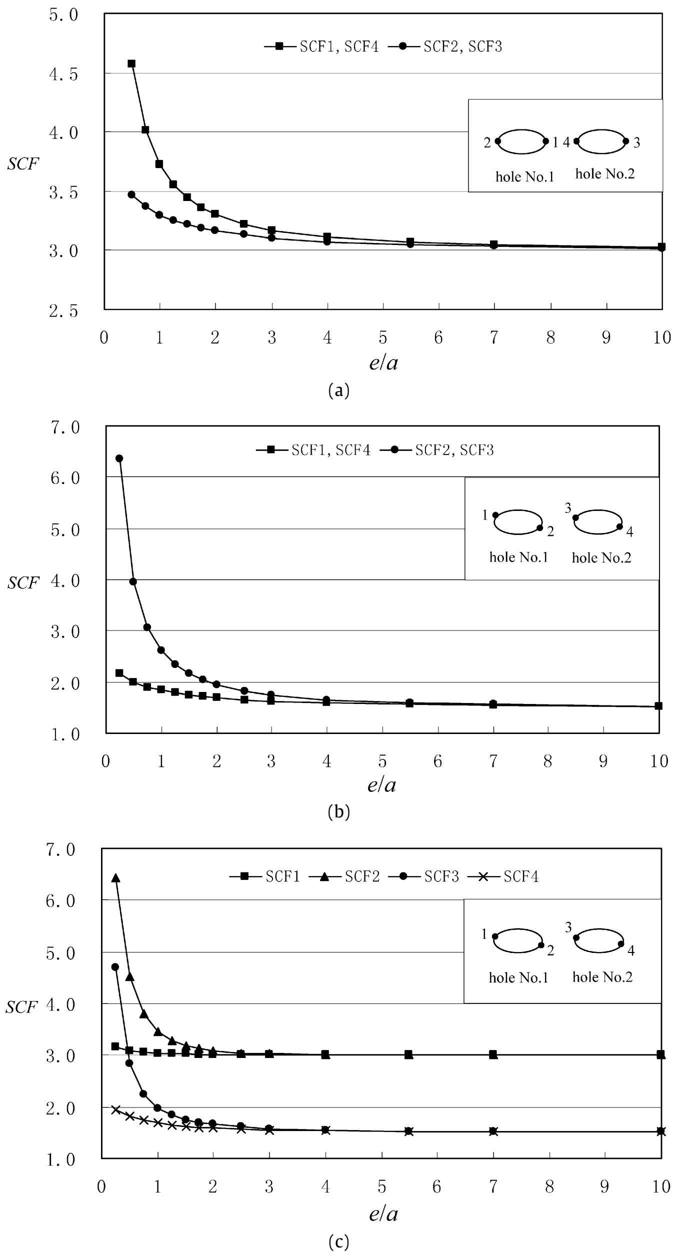

Fig. 5. Variation of stress concentrations with separation distance e/a on the boundaries of the two holes, under three loading cases. (a) For the case of $N_1 = N_2 = -1$ and $S_1 = S_2 = 0$. (b) For the case of $N_1 = N_2 = 0$ and $S_1 = S_2 = 1$. (c) For the case of $-N_1 = S_2 = 1$ and $N_2 = S_1 = 0$.

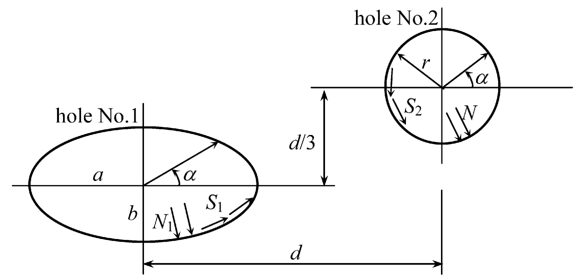

Fig. 6. An elliptic hole ($a/b = 2$) and a circular hole ($r = b$) in an infinite elastic plate subjected to uniform normal pressures N_1, N_2 (negative) and tangential shears S_1 and S_2 (positive) on the hole boundaries.

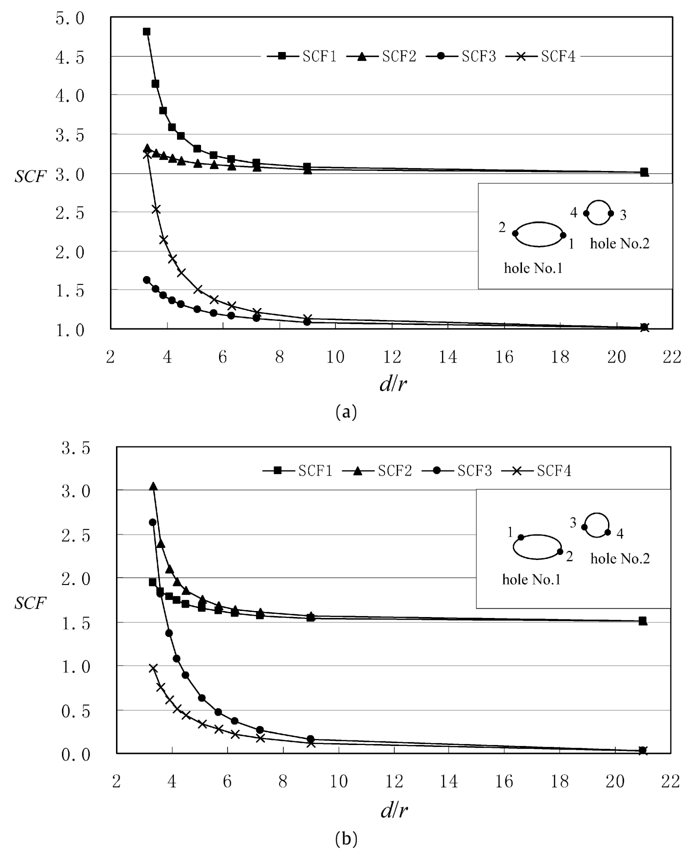

Fig. 7. Variation of stress concentrations with the center-to-center distance d/r on the boundaries of the two holes under two loading cases. (a) For the case of $N_1 = N_2 = -1$ and $S_1 = S_2 = 0$. (b) For the case of $N_1 = N_2 = 0$ and $S_1 = S_2 = 1$.

The stress concentration factors for the two loading cases are defined as $SCF = \sigma_{\theta\,\max}/|N_1|$ and $SCF = \sigma_{\theta\,\max}/S_1$, respectively. The four stress concentrations, i.e., SCF1, SCF2, SCF3 and SCF4, have the same definitions as those in Section 4.1, with their position numbers plotted in Fig. 7 (a) and (b).

Fig. 7(a) provides the stress concentrations for the case (1). As the distance d/r increases, SCF1 and SCF2 gradually approach to 3.0, and SCF3 and SCF4 to zeroes. Moreover, the four stress concentrations, SCF1 and SCF2, SCF3 and SCF4, have various positions of from 4.48 to 0.05 degrees, from 180.25 to 180.00 degrees along the hole No. 1, from 6.80 to 1.40 degrees and from 194.60 to 181.00 degrees along the hole No. 2, respectively.

Fig. 7(b) provides the stress concentration factors for the case (2). As the distance d/r increases, SCF1 and SCF2 gradually decrease to 1.5, and SCF3 and SCF4 to zeroes. Moreover, the four stress concentrations, SCF1 and SCF2, SCF3 and SCF4, have various positions of from 165.83 to 166.00 degrees, from 358.45 to 346.00 degrees along the hole No. 1, from 189.00 to 164.65 degrees and from 311.80 to 338.10 degrees along the hole No. 2, respectively.

5. Conclusions

Using the Schwarz's alternating method and the Muskhelishvili's complex variable function techniques, an iterative algorithm is presented to obtain accurate stresses in a plate containing two elliptic holes subjected to uniform normal tensions and tangential shears on the hole boundaries and any uniform loads applied at infinity. The iterative algorithm can be used to obtain accurate stress solutions for the double elliptic hole problems. However, when two holes are arranged by both smaller separation distance (e.g., $e/a < 0.5$ in Fig. 4) and bigger size difference (e.g., the hole No. 1 is 10 times bigger than the hole No. 2 in Fig. 4), the algorithm should be further improved so as to increase efficiency and accuracy of iterative stress solution for some extreme cases.

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