GEOPHYSICS, VOL. 81, NO. 5 (SEPTEMBER-OCTOBER 2016); P. C219–C227, 5 FIGS.
10.1190/GEO2015-0094.1

Long-offset moveout for VTI using Padé approximation

Hanjie Song[1], Yingjie Gao[1], Jinhai Zhang[2], and Zhenxing Yao[2]

ABSTRACT

The approximation of normal moveout is essential for estimating the anisotropy parameters of the transversally isotropic media with vertical symmetry axis (VTI). We have approximated the long-offset moveout using the Padé approximation based on the higher order Taylor series coefficients for VTI media. For a given anellipticity parameter, we have the best accuracy when the numerator is one order higher than the denominator (i.e., $[L/(L-1)]$); thus, we suggest using [4/3] and [7/6] orders for practical applications. A [7/6] Padé approximation can handle a much larger offset and stronger anellipticity parameter. We have further compared the relative traveltime errors between the Padé approximation and several approximations. Our method shows great superiority to most existing methods over a wide range of offset (normalized offset up to 2 or offset-to-depth ratio up to 4) and anellipticity parameter (0–0.5). The Padé approximation provides us with an attractive high-accuracy scheme with an error that is negligible within its convergence domain. This is important for reducing the error accumulation especially for deeper substructures.

INTRODUCTION

The anisotropy phenomena in wave propagations have been widely recognized on various scales. The presence of anisotropy leads to nonhyperbolic moveout even in a homogeneous layer. A transversely isotropic medium with a vertical symmetry axis (VTI) is a reasonable approximation of horizontally layered anisotropic medium. The traveltime approximation is essential for determining the anisotropy parameters of layered VTI media, and thus it gains a lot of attention. Tsvankin and Thomsen (1994, 1995) generalize the three-term equation (Hake et al., 1984) by including the fourth-order Taylor coefficient for long offset in VTI media. Alkhalifah and Tsvankin (1995) rewrite the generalized three-term equation in terms of normal moveout (NMO) velocity v_{NMO} and the anellipticity parameter η. Alkhalifah (1997) generalizes the Dix (1955) equation by introducing effective anellipticity η_{eff} to multilayered VTI media. Fowler (2003) summarizes a comprehensive list of traveltime approximations in VTI media for the qP-wave. Stovas and Ursin (2004) propose a continued-fraction approximation for a single VTI layer. Fomel (2004) and Fomel and Stovas (2010) propose two general forms of the nonhyperbolic approximation that can be applied to any kind of media. Ursin and Stovas (2006) point out that approximations of traveltime squared are generally better than those for traveltime (Malovichko, 1978; de Bazelaire, 1988; Castle, 1994; Siliqi and Bousquie, 2000; Fomel, 2004). Blias (2009) and Golikov and Stovas (2012) perform an extensive comparison of traveltime approximations for VTI media. Ravve and Koren (2009, 2010, 2015) derive an accurate equation for hyperbolic and high-order terms of the NMO in the presence of a package of horizontal symmetry axis/VTI layers or layered orthorhombic media. Aleixo and Schleicher (2010) propose some new single-parameter traveltime approximations based on the approximations found in the literature.

Most existing moveout approximations, either using traveltime or traveltime squared (Ursin and Stovas, 2006), are still not accurate enough for a single VTI layer, although various forms have been presented using either Taylor expansion or continued-fraction approximation. In mathematics, as a powerful approach derived from the Taylor expansion, the Padé expansion has proved to be superior to the Taylor expansion and the continued-fraction approximation (Baker, 1975; Ma, 2005; Aptekarev et al., 2011). It is the "best" approximation of a function by a rational function for a given order. The Padé approximation may still work even when the Taylor expansion does not converge. Douma and Calvert (2006) and Douma and van der Baan (2008) build an accurate moveout approximation by using Padé interpolation or rational interpolation between traveltimes from several rays.

Manuscript received by the Editor 9 February 2015; revised manuscript received 10 April 2016; published online 5 August 2016.
[1]Institute of Geology and Geophysics, Key Laboratory of Earth and Planetary Physics, Chinese Academy of Sciences, Beijing, China and University of Chinese Academy of Sciences, Beijing, China. E-mail: songhj@mail.iggcas.ac.cn; gaoyingjie@mail.iggcas.ac.cn.
[2]Institute of Geology and Geophysics, Key Laboratory of Earth and Planetary Physics, Chinese Academy of Sciences, Beijing, China. E-mail: zjh@mail.iggcas.ac.cn; yaozx@mail.igcas.ac.cn.

In this paper, we present an alternative moveout approximation using the Padé expansion directly. We suggest two groups of equations, [4/3] and [7/6] order Padé approximations, based on the traveltime squared for VTI layer. Theoretical analysis and numerical experiments show that our method is superior to many existing approximations within a wide range of anellipticity parameter, especially for a large normalized offset or offset to depth ratio.

EXACT TRAVELTIME FOR SINGLE VTI LAYER

We consider qP-wave propagation in a homogeneous VTI layer. It is convenient to define the nondimensional anisotropy parameters in terms of elastic tensor components (Thomsen, 1986)

$$\alpha_0 = \sqrt{\frac{c_{33}}{\rho}},\quad \beta_0 = \sqrt{\frac{c_{55}}{\rho}},\quad \varepsilon = \frac{c_{11}-c_{33}}{2c_{33}},\quad \delta = \frac{(c_{13}+c_{55})^2-(c_{33}-c_{55})^2}{2c_{33}(c_{33}-c_{55})}, \tag{1}$$

where α_0 and β_0 are the vertical velocities of the qP- and qS-waves, respectively, and ε and δ are the dimensionless anisotropy parameters (Thomsen, 1986), respectively. It is well-known that the qP-wave phase velocity in transversally isotropic media depends weakly on β_0 (Tsvankin and Thomsen, 1994; Alkhalifah, 1998). We set $\beta_0 = 0$, which is the acoustic approximation from Alkhalifah (1998, 2000). Therefore, the phase velocity reads (Tsvankin, 2001)

$$v_p(\theta) = \alpha_0\sqrt{\varepsilon\sin^2\theta+\frac{1}{2}\left(1+\sqrt{(1+2\varepsilon\sin^2\theta)^2-2(\varepsilon-\delta)\sin^2 2\theta}\right)}, \tag{2}$$

where v_p is the phase velocity and θ is the phase angle. The group angle and the group velocity are related to the phase angle and phase velocity through (Tsvankin, 2001)

$$\tan\psi = \frac{\tan\theta + \frac{1}{v_p(\theta)}\frac{dv_p}{d\theta}}{1-\frac{\tan\theta}{v_p(\theta)}\frac{dv_p}{d\theta}}, \tag{3}$$

$$v_g = v_p(\theta)\sqrt{1+\left(\frac{1}{v_p(\theta)}\frac{dv_p}{d\theta}\right)^2}, \tag{4}$$

where v_g is the group velocity and ψ is the group angle. The normalized traveltime reads (Ursin and Stovas, 2006)

$$\tau = \frac{t}{t_0} = \frac{v_{\mathrm{NMO}}}{v_g\cos\psi}, \tag{5}$$

and the normalized offset reads

$$x = \frac{X}{t_0 v_{\mathrm{NMO}}} = \frac{v_0\tan\psi}{v_{\mathrm{NMO}}}, \tag{6}$$

where v_{NMO} is the NMO velocity defined as $v_{\mathrm{NMO}} = \alpha_0\sqrt{1+2\delta}$, t is the traveltime, and X is the offset.

TAYLOR SERIES OF TRAVELTIME SQUARED

For a homogeneous VTI layer, the Taylor series coefficients are well-known (Golikov and Stovas, 2012) and can be computed from the parametric offset-traveltime equations. The normalized traveltime $\tau(p)$ and the normalized offset $x(p)$ are as follows (Ursin and Stovas, 2006):

$$\tau(p) = \sum_{n=0}^{\infty} b_n\mu_{2n}p^{2n}, \tag{7}$$

$$x(p) = \frac{p}{v_{\mathrm{NMO}}}\sum_{n=0}^{\infty} b_n\mu_{2n+2}p^{2n}, \tag{8}$$

where

$$b_n = \begin{cases} 1, & n=0, \\ \frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2^n n!}, & n=1,2,\ldots \end{cases} \tag{9}$$

and p is the horizontal slowness (the horizontal component of the slowness vector; also called the ray parameter). According to Ursin and Stovas (2006), the coefficients μ_{2k} are

$$\mu_{2k} = \begin{cases} 1, & k=0, \\ \frac{a_{k-1}k\alpha_0^{2k}+\sum_{j=0}^{k-2}(a_j\alpha_0^{2j+2}\mu_{2(k-1)-2j}((j+1)b_{k-j-1}-jb_{k-j-2}))}{b_{k-1}}, & k=1,2,\ldots \end{cases} \tag{10}$$

for qPqP reflection in a single layer, where the coefficient α_0 is the vertical qP-wave velocity and

$$a_n = 2^n(1+2\delta)(\varepsilon-\delta)^n,\quad n=0,1,2,\ldots \tag{11}$$

The derivation of a Taylor series for $\tau^2(x)$ is exactly as in Taner and Koehler (1969) or Hubral and Krey (1980); that is,

$$\tau^2(x) = 1 + x^2 + c_2x^4 + c_3x^6 + c_4x^8 + \cdots. \tag{12}$$

Squaring equations 7 and 8, we have

$$\tau^2(p) = \sum_{n=0}^{\infty} R_n p^{2n}, \tag{13}$$

$$x^2(p) = p^2\sum_{n=0}^{\infty} B_{n,0}p^{2n}, \tag{14}$$

where

$$R_n = \sum_{i=0}^{n} b_i \mu_{2i} b_{n-i} \mu_{2(n-i)} \quad (15)$$

and

$$B_{n,0} = \frac{1}{v_{\mathrm{NMO}}^2} \sum_{i=0}^{n} b_i \mu_{2i+2} b_{n-i} \mu_{2(n-i)+2}. \quad (16)$$

Let

$$x^{2k}(p) = p^{2k} \sum_{n=0}^{\infty} B_{n,k-1} p^{2n}, \quad k = 1,2, \ldots, \quad (17)$$

then, the coefficients $B_{n,k}$ can be determined recursively by

$$B_{n,k} = \sum_{k=1}^{\infty} \sum_{i=0}^{n} B_{n-i,k-1} B_{i,0}, \quad k = 1,2,\ldots; n = 0,1,\ldots \quad (18)$$

Substituting equations 13 and 17 into equation 12 and equating coefficients, we have

$$R_n = \sum_{i=1}^{n} c_{n-i+1} B_{i-1,n-i}, \quad n = 1,2, \ldots \quad (19)$$

Explicit formulas for c_2, c_3, and c_4 are given in terms of the coefficients μ_{2k} as follows:

$$\begin{aligned} c_2 &= \frac{\mu_2^2 - \mu_4}{4\mu_2^2}, \\ c_3 &= \frac{2\mu_4^2 - \mu_2^2\mu_4 - \mu_2\mu_6}{8\mu_2^4}, \\ c_4 &= \frac{24\mu_2\mu_4\mu_6 + 9\mu_2^2\mu_4^2 - 24\mu_4^3 - 4\mu_2^3\mu_6 - 5\mu_2^2\mu_8}{64\mu_2^6}, \end{aligned} \quad (20)$$

where μ_2, μ_4, μ_6, and μ_8 are calculated by equation 10 as

$$\begin{aligned} \mu_2 &= v_{\mathrm{NMO}}^2, \\ \mu_4 &= (1 + 8\eta) v_{\mathrm{NMO}}^4, \\ \mu_6 &= (1 + 8\eta + 32\eta^2) v_{\mathrm{NMO}}^6, \\ \mu_8 &= \left(1 + \frac{48}{5}\eta + \frac{192}{5}\eta^2 + \frac{512}{5}\eta^3\right) v_{\mathrm{NMO}}^8, \end{aligned} \quad (21)$$

where η is the anellipticity parameter defined as $\eta = (\varepsilon - \delta)/(1 + 2\delta)$. The coefficients c_k in a single layer can be described only by anellipticity parameter η as follows:

$$\begin{aligned} c_2 &= -2\eta, \\ c_3 &= 2\eta(1 + 6\eta), \\ c_4 &= -2\eta(1 + 16\eta + 52\eta^2). \end{aligned} \quad (22)$$

Therefore, the Taylor series for $\tau^2(x)$ in a single layer consists of the terms that include offset powers, where the coefficients of these terms depend on the intrinsic anellipticity parameter η. On the basis of higher order coefficients of the Taylor series, we use the Padé expansion to construct the traveltime approximation for a single VTI layer. The first 15 coefficients of the Taylor series are listed in Appendix A.

For the conveniences of numerical comparison, we list several well-known methods below. Conventional Dix-type layer stripping using the nonhyperbolic approximation, suggested by Tsvankin and Thomsen (1994) and modified by Alkhalifah and Tsvankin (1995), can be rewritten as

$$\tau^2(x) = 1 + x^2 - \frac{2\eta x^4}{1 + (1 + 2\eta)x^2}. \quad (23)$$

Based on the moveout equation for the layered media (Castle, 1994), Siliqi (2001) proposes a shifted hyperbola approximation for the traveltime in terms of offset

$$\tau(x) = 1 + \frac{1}{1 + 8\eta}\left[\sqrt{1 + (1 + 8\eta)x^2} - 1\right]. \quad (24)$$

Ursin and Stovas (2006) propose a generalized continued-fraction traveltime approximation for VTI media and match the approximation to the Taylor expansion up to the order x^6 (i.e., equations 18 and 19 in their paper)

$$\tau^2(x) = 1 + x^2 - \frac{2\eta x^4}{1 + Bx^2}, \quad (25)$$

where $B = 1 + 6\eta$ for a single layer (Golikov and Stovas, 2012). Fomel and Stovas (2010) propose the general form of the moveout approximation

$$\tau^2(x) = 1 + x^2 - \frac{4\eta x^4}{1 + \frac{1+8\eta+8\eta^2}{1+2\eta}x^2 + \sqrt{1 + \frac{2(1+8\eta+8\eta^2)}{1+2\eta}x^2 + \frac{x^4}{(1+2\eta)^2}}}, \quad (26)$$

which can be applied to any type of media. This function is identical to the approximation proposed by Fomel (2004).

PADÉ APPROXIMATION

The Padé approximation is a rational function based on the power series expansion that agrees with a prescribed power series to the highest possible order (Baker, 1975; Ma, 2005; Aptekarev et al., 2011). For a given power series

$$f(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k, \quad (27)$$

the rational function

$$R_{LM}(\lambda) = \frac{P_L(\lambda)}{Q_M(\lambda)} = \frac{\sum_{k=0}^{L} P_k \lambda^k}{\sum_{k=0}^{M} Q_k \lambda^k} \quad (28)$$

is defined as $[L/M]$ order Padé approximation if

$$f(\lambda) - R_{LM}(\lambda) = O(\lambda^{L+M+1}). \quad (29)$$

The normalization condition $Q_0 = 1$ is used to solve the system; thus, the approximation is just the Maclaurin expansion for $f(\lambda)$. We write the coefficients of $P_L(\lambda)$ and $Q_M(\lambda)$ as

$$\begin{cases} P_L(\lambda) = P_0 + P_1\lambda + \cdots + P_L\lambda^L, \\ Q_M(\lambda) = 1 + Q_1\lambda + \cdots + Q_M\lambda^M. \end{cases} \quad (30)$$

Equation 29 is equivalent to the requirements

$$f(0) = R_{LM}(0); \frac{d^k}{d\lambda^k} f(\lambda)|_{\lambda=0} = \frac{d^k}{d\lambda^k} R_{LM}(\lambda)|_{\lambda=0}, \quad k = 1,2, \ldots, L+M. \quad (31)$$

Therefore, $f(\lambda)$ and $R_{LM}(\lambda)$ agree at $\lambda = 0$ and their derivatives up to $L + M$ agree at $\lambda = 0$.

The first $L + M + 1$ terms of $R_{LM}(\lambda)$ match the first $L + M + 1$ terms of the power series of $f(\lambda)$. Thus, the coefficients $P_k(k = 0, \ldots, L)$ and $Q_k(k = 1, \ldots, M)$ can be obtained by multiplying the denominator of $R_{LM}(\lambda)$ by equation 29 and equating coefficients of λ^k with $k = 0, \ldots, L + M$. The result is

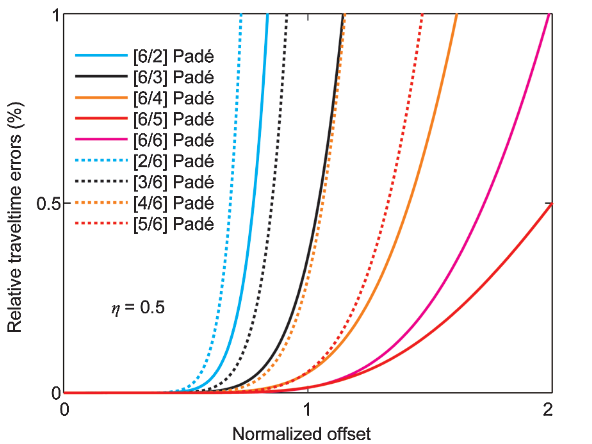

Figure 1. Comparison of relative traveltime errors between various sixth-order Padé approximations when $\eta = 0.5$.

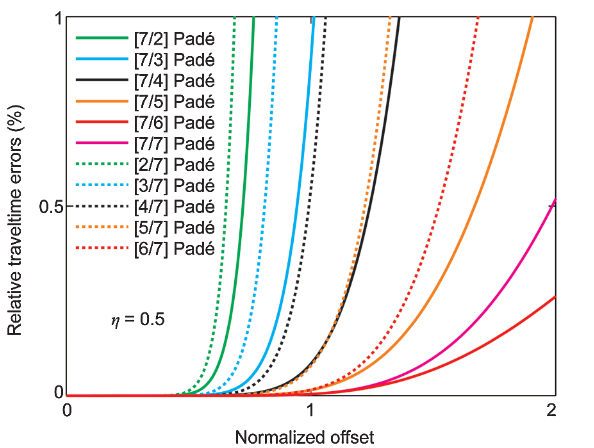

Figure 2. Comparison of relative traveltime errors between various seventh-order Padé approximations when $\eta = 0.5$.

$$\begin{cases} \sum_{i=0}^{M} Q_i c_{k-i} = P_k, & k = 0, \ldots, L, \\ \sum_{i=1}^{M} Q_i c_{k-i} = 0, & k = L+1, \ldots, L+M, \end{cases} \quad (32)$$

where c_k is the kth coefficient in the Taylor series, as shown in Appendix A. Equation 32 can be rewritten as

$$\begin{cases} c_0 = P_0, \\ c_1 + c_0 Q_1 = P_1, \\ c_2 + c_1 Q_1 + c_0 Q_2 = P_2, \\ \vdots \\ c_L + c_{L-1}Q_1 + \cdots + c_{L-M}Q_M = P_L, \\ c_{L+1} + c_L Q_1 + \cdots + c_{L-M+1}Q_M = 0, \\ \vdots \\ c_{L+M} + c_{L+M-1}Q_1 + \cdots + c_L Q_M = 0. \end{cases} \quad (33)$$

The coefficient of c_0 is always equal to one with normalized offset and normalized traveltime; thus, $P_0 = 1$. With $\lambda \equiv x^2$, we can solve the coefficients P_k and Q_k for the Taylor series $\tau^2(x)$ in function 12, and the $[L/M]$-order Padé approximation is

$$\tau_{LM}^2(x) = R_{LM}(\lambda) = \frac{1 + \sum_{k=1}^{L} P_k x^{2k}}{1 + \sum_{k=1}^{M} Q_k x^{2k}}. \quad (34)$$

We select the optimal order of the Padé approximation by numerical tests. For a typical anellipticity parameter η, the best accuracy arises when $M = L - 1$; furthermore, the accuracy is stably increasing when M approaches $L - 1$, as shown in Figures 1, 2, and 3. From the form of the approximation suggested by Alkhalifah and Tsvankin (1995), we can see that the continued-fraction approximation is one order higher in the numerator than in the denominator, which is consistent with our selection $M = L - 1$. To preserve the asymptotic behavior for a homogeneous VTI medium, the coefficients P_k and Q_k should satisfy certain relations. To preserve the horizontal velocity, $Q_{L-1}/P_L = 1 + 2\eta$, and to preserve the intercept for asymptote, $(P_{L-1}Q_{L-1} - P_L Q_{L-2})/Q_{L-1}^2 = 1 + 2\eta$, as shown in Appendix B. The coefficients of [4/3] and [7/6] orders are shown in Appendixes C and D, respectively.

NUMERICAL EXPERIMENTS

Blias (2009) tests several three-term moveout approximations by a velocity model with strong anisotropy at long offset. In fact, his strong anellipticity parameter η ranges from 0 to 0.33; in addition, Blias's largest offset to depth ratio (O/D) is only approximately 2 (around normalized offset of one), which is an intermediate case defined by Tsvankin and Thomsen (1994). The accuracy of the different traveltime approximations is compared by the qPqP reflection data generated for a single-layer VTI model. The depth of the model is 1000 m, and the vertical velocity is 2000 m/s. The theoretical normalized traveltime is computed by equations 5 and 6.

Figures 1 and 2 show the various sixth- and seventh-order Padé approximations when $\eta = 0.5$, respectively. Obviously, when the orders for either numerator or denominator are too far away from the diagonal order (i.e., $L = M = 6$ or 7), the accuracy would decrease rapidly for the given anellipticity parameter η. For example, as shown in Figure 1, the accurate normalized offset range for the [2/6] or [6/2] Padé approximations is only approximately 0.5; in

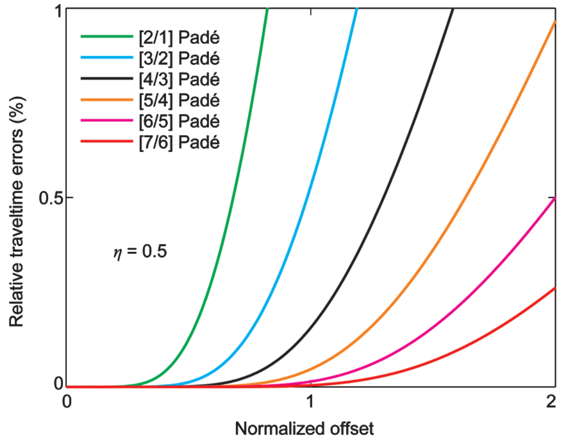

Figure 3. Comparison of relative traveltime errors between $[L/(L-1)]$ Padé approximations ($L = 2, 3, \ldots, 7$) when $\eta = 0.5$.

contrast, that for the [6/5] or [6/6] Padé approximations is bigger than one. Meanwhile, we can see that [6/5] is superior to [5/6]; moreover, [6/5] is even superior to [6/6]. Similarly, [7/6] is superior to [6/7] and even [7/7], as shown in Figure 2. This indicates that we should use a higher order of the numerator than that of the denominator to obtain the optimal accuracy of traveltime given a limited order of the Padé approximation. In fact, the continued-fraction approximation for the VTI media is basically one order higher in the numerator than in the denominator. This fact is consistent with our results. Therefore, we suggest using $[L/(L-1)]$ Padé approximations.

Figure 3 shows the relative traveltime errors of various $[L/(L-1)]$ Padé approximations ($L = 2, 3, \ldots, 7$) when $\eta = 0.5$. Obviously, we should use high orders because they can provide us an apparently wider accurate normalized offset range. For example, the accurate normalized offset range for the [2/1] Padé approximation is smaller than 0.5, and that for [4/3] is bigger than 0.5; in contrast that for [7/6] is bigger than one. Therefore, we use the [4/3] and [7/6] Padé approximations for further comparison in our numerical experiments.

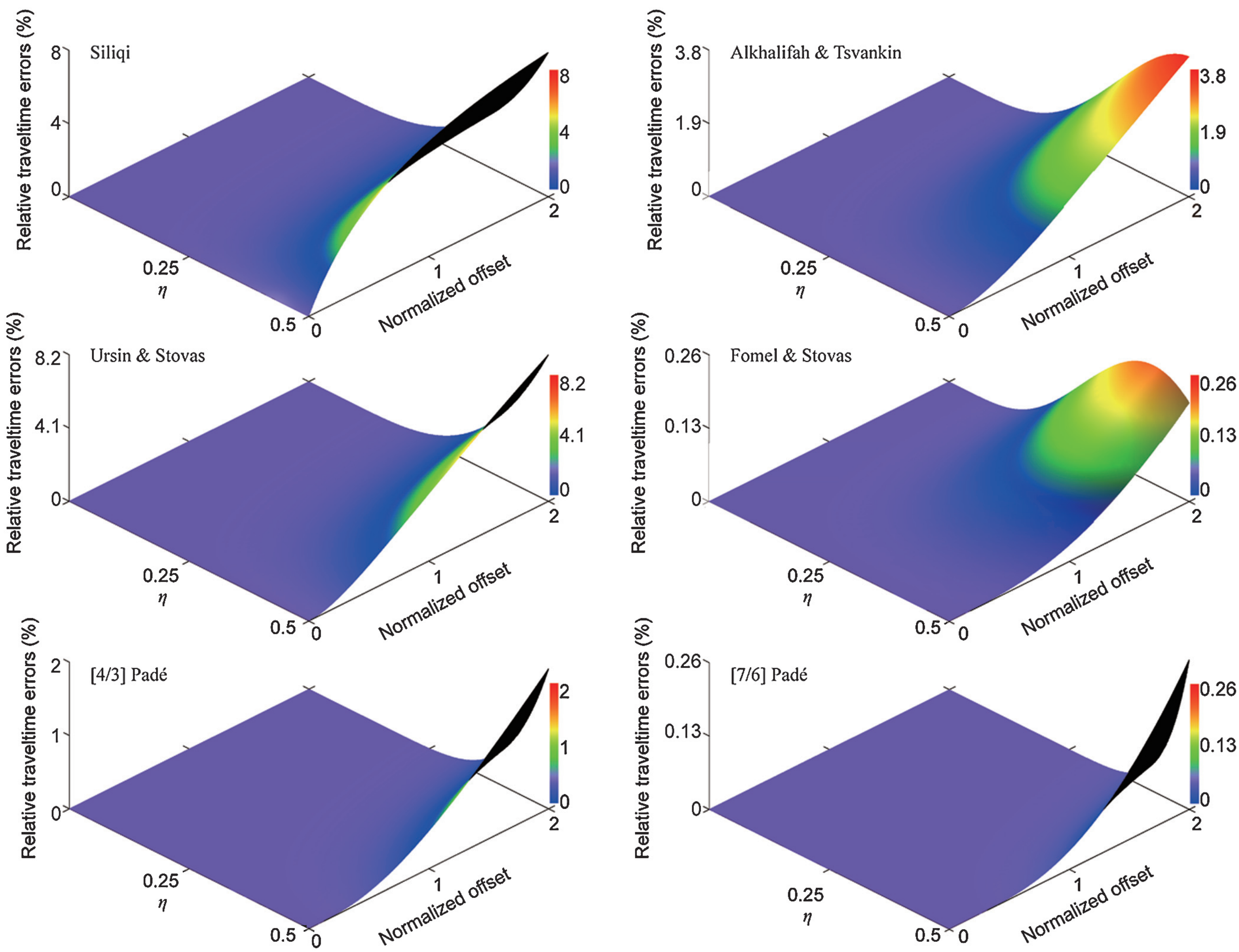

Figure 4. The 3D plots of relative traveltime errors for different approximations (Alkhalifah and Tsvankin, 1995; Siliqi, 2001; Ursin and Stovas, 2006; Fomel and Stovas, 2010; [4/3] and [7/6] Padé approximations). We show $0 < \eta < 0.5$ with an interval of 0.01 when normalized offset $x < 2$.

Figure 4 shows 3D plots of relative traveltime errors for different approximations (Alkhalifah and Tsvankin, 1995; Siliqi, 2001; Ursin and Stovas, 2006; Fomel and Stovas, 2010; [4/3] and [7/6] Padé approximations). We show $0 < \eta < 0.5$ with an interval of 0.01 when normalized offset $x < 2$ (i.e., $O/D \approx 4.5$). Obviously, when the anellipticity parameter η is close to zero, all approximations listed are accurate enough. However, when η is far away from zero, most approximations listed show apparent errors. In contrast, our Padé approximation has similar error within a much wider normalized offset or O/D range. The [7/6] order Padé approximation is comparable with the approximation by Fomel and Stovas (2010) for most cases, but it is slightly better in case of relatively big η and normalized offset x. Approximately $\eta = 0.5$ and $x = 2$, the [7/6] order Padé approximation has slightly larger error than the approximation by Fomel and Stovas (2010).

Figure 5 shows the comparison between all methods in the Greenhorn shale (Jones and Wang, 1981), which is also used by Fomel (2004). In this medium, we have $\varepsilon = 0.256$, $\delta = -0.0505$, and $\eta = 0.3409$. Obviously, most approximations listed show apparent errors, especially for the sixth-order Taylor approximation due to inherent defects of Taylor expansion. The approximation by Fomel and Stovas (2010) and the Padé approximations do not exceed relative error of 1% when $x < 2$. The [7/6] Padé approximation has similar range of relative error to the approximation by Fomel and Stovas (2010), but with slightly smaller error.

DISCUSSION

It is difficult for the inversion based on the Dix-type approximation to yield exact interval vertical velocities for multilayer structures due to the tradeoff between the NMO velocity and anisotropy parameters (Alkhalifah and Tsvankin, 1995; Alkhalifah, 1997). In addition, Dix-type layer stripping suffers from serious error accumulation. As a powerful alternative to the generalized Dix-type layer stripping, the velocity-independent layer stripping (Dewangan and Tsvankin, 2006; Wang and Tsvankin, 2009) is more stable in layer stripping of reflection traveltime, where the interval traveltime is converted into single-layer forms. Besides, we can also apply the parametric offset-traveltime equations (Fomel and Stovas, 2010; Stovas, 2010) to extend the proposed Padé approximation to multilayered medium, whereas the asymptotic behavior would be complicated because the Padé approximation involves a large number of coefficients.

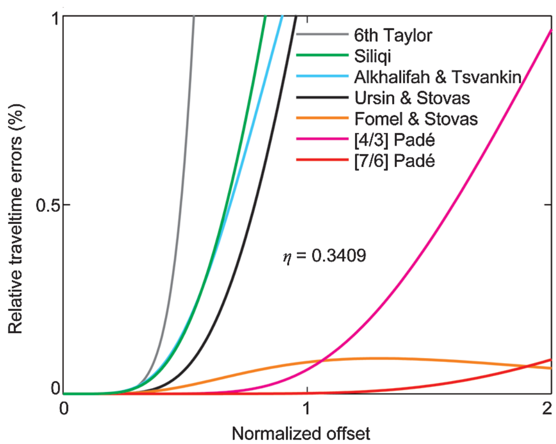

Figure 5. Relative traveltime errors of different approximations for Greenhorn shale (sixth-order Taylor; Alkhalifah and Tsvankin, 1995; Siliqi, 2001; Ursin and Stovas, 2006; Fomel and Stovas, 2010; [4/3] and [7/6] Padé approximations) when normalized offset $x < 2$.

CONCLUSION

The analysis of the anellipticity parameter is significantly sensitive to the traveltime; thus, we should guarantee that the relative traveltime error is as small as possible. We present the Padé approximation of traveltime squared for the nonhyperbolic moveout of the qP-wave in VTI media with a strong anellipticity parameter. This method is a simple approach to directly use large terms of the Taylor series by Padé approximation. Numerical experiments show that the Padé approximation can handle a much larger offset and a stronger anellipticity parameter than most existing approximations. The Padé approximation provides us with an attractive scheme, whose error is negligible within its convergence domain. Our method has a much smaller traveltime error within the most commonly used range of anisotropy and the normalized offset (i.e., $0 < \eta < 0.5$ and $x < 2$), which is important for reducing the error accumulation, especially for deeper substructures.

ACKNOWLEDGMENTS

This research was supported by the National Natural Science Foundation of China (41130418 and 41374061) and the National Major Project of China (2011ZX05008-006). J. Zhang was also supported by the Youth Innovation Promotion Association CAS (2012054). We are grateful to A. Stovas and I. Ravve for giving us many helpful comments and insightful suggestions that greatly improve the manuscript.

APPENDIX A

COEFFICIENTS OF TAYLOR SERIES

$$c_0 = 1,$$

$$c_1 = 1,$$

$$c_2 = -2\eta,$$

$$c_3 = 2\eta(1+6\eta),$$

$$c_4 = -2\eta(1+16\eta+52\eta^2),$$

$$c_5 = 2\eta(1+30\eta+240\eta^2+544\eta^3),$$

$$c_6 = -2\eta(1+48\eta+684\eta^2+3648\eta^3+6384\eta^4),$$

$$c_7 = 2\eta(1+70\eta+1540\eta^2+14168\eta^3+56672\eta^4+80960\eta^5),$$

$$c_8 = -2\eta(1+96\eta+3000\eta^2+41600\eta^3+280800\eta^4+898560\eta^5+1085760\eta^6),$$

$$c_9 = 2\eta(1+126\eta+5292\eta^2+102312\eta^3+1023120\eta^4+5437152\eta^5 \\ +14499072\eta^6+15189504\eta^7),$$

$$c_{10} = -2\eta\begin{pmatrix}1+160\eta+8680\eta^2+222208\eta^3 \\ +3055360\eta^4+23744512\eta^5+103882240\eta^6 \\ +237445120\eta^7+219636736\eta^8\end{pmatrix},$$

$$c_{11}=2\eta\begin{pmatrix}1+198\eta+13464\eta^2+439824\eta^3\\+7916832\eta^4+83692224\eta^5+530050752\eta^6\\+1968759936\eta^7+3937519872\eta^8+3261380096\eta^9\end{pmatrix},$$

$$c_{12}=-2\eta\begin{pmatrix}1+240\eta+19980\eta^2+809856\eta^3\\+18424224\eta^4+252675072\eta^5+2158266240\eta^6\\+11510753280\eta^7+37122179328\eta^8+65994985472\eta^9\\+49496239104\eta^{10}\end{pmatrix},$$

$$c_{13}=2\eta\begin{pmatrix}1+286\eta+28600\eta^2+1407120\eta^3\\+39399360\eta^4+677668992\eta^5+7454358912\eta^6\\+53245420800\eta^7+244928935680\eta^8+697676362240\eta^9\\+1116282179584\eta^{10}+765004570624\eta^{11}\end{pmatrix},$$

$$c_{14}=-2\eta\begin{pmatrix}1+336\eta+39732\eta^2+2330944\eta^3\\+78669360\eta^4+1654304256\eta^5+22677754176\eta^6\\+207339466752\eta^7+1269954233856\eta^8+5131128217600\eta^9\\+13084376954880\eta^{10}+19031821025280\eta^{11}+12008172789760\eta^{12}\end{pmatrix}. \quad \text{(A-1)}$$

APPENDIX B

VERIFYING THE RELATIONS BETWEEN P_k AND Q_k OF $[L/(L-1)]$ ORDERS

For [4, 3] Padé, the relations are

$$Q_{L-1}/P_L = 1 + 2\eta - 216\eta^4 + O[\eta^5] \quad \text{(B-1)}$$

and

$$(P_{L-1}Q_{L-1}-P_L Q_{L-2})/Q_{L-1}^2 = 1+2\eta-1512\eta^4+O[\eta^5] \quad \text{(B-2)}$$

for [7/6] Padé,

$$Q_{L-1}/P_L = 1 + 2\eta - 279936\eta^7/7 + O[\eta^8] \quad \text{(B-3)}$$

and

$$(P_{L-1}Q_{L-1}-P_L Q_{L-2})/Q_{L-1}^2 = 1+2\eta-3639168\eta^7/7+O[\eta^8]. \quad \text{(B-4)}$$

APPENDIX C

COEFFICIENTS OF [4/3] PADÉ APPROXIMATION

$$P_1=\frac{20+359\eta+2376\eta^2+6904\eta^3+7432\eta^4}{5+59\eta+236\eta^2+316\eta^3},$$

$$P_2=\frac{2(15+334\eta+2939\eta^2+12760\eta^3+27320\eta^4+23072\eta^5)}{5+59\eta+236\eta^2+316\eta^3},$$

$$P_3=\frac{20+495\eta+5044\eta^2+26692\eta^3+76488\eta^4+110992\eta^5+62304\eta^6}{5+59\eta+236\eta^2+316\eta^3},$$

$$P_4=\frac{5+127\eta+1356\eta^2+7676\eta^3+24088\eta^4+39504\eta^5+26336\eta^6}{5+59\eta+236\eta^2+316\eta^3}, \quad \text{(C-1)}$$

$$Q_1=\frac{15+300\eta+2140\eta^2+6588\eta^3+7432\eta^4}{5+59\eta+236\eta^2+316\eta^3},$$

$$Q_2=\frac{15+378\eta+3856\eta^2+19404\eta^3+47840\eta^4+46144\eta^5}{5+59\eta+236\eta^2+316\eta^3},$$

$$Q_3=\frac{5+137\eta+1610\eta^2+10388\eta^3+38360\eta^4+75920\eta^5+62304\eta^6}{5+59\eta+236\eta^2+316\eta^3}. \quad \text{(C-2)}$$

APPENDIX D

COEFFICIENTS OF [7/6] PADÉ APPROXIMATION

Defining

$$A=\begin{pmatrix}137781+7873200\eta+210958020\eta^2+3516143904\eta^3\\+40771048680\eta^4+348378661728\eta^5+2266037591552\eta^6\\+11424068883456\eta^7+44998587356928\eta^8+138452714795520\eta^9\\+329947131867648\eta^{10}+597877043361792\eta^{11}\\+797105482070016\eta^{12}+737865270116352\eta^{13}\\+423872407166976\eta^{14}+113863061831680\eta^{15}\end{pmatrix}, \quad \text{(D-1)}$$

we have

$$P_1=\frac{\begin{pmatrix}1928934+125164197\eta+3804702030\eta^2+71969817384\eta^3\\+948741278976\eta^4+9246135527424\eta^5+68935042487488\eta^6\\+401166352961472\eta^7+1841917418277504\eta^8+6694990617534208\eta^9\\+19200531023097344\eta^{10}+42986528530983936\eta^{11}\\+73639779954829312\eta^{12}+93296259238731776\eta^{13}\\+82419789374570496\eta^{14}+45348731451998208\eta^{15}\\+11703985675108352\eta^{16}\end{pmatrix}}{2A},$$

$$P_2=\frac{\begin{pmatrix}2893401+206336889\eta+6920509995\eta^2+145009718802\eta^3\\+2126469269592\eta^4+23164618395744\eta^5+194149518536832\eta^6\\+1279023894214208\eta^7+6706005433547200\eta^8+28145390724455040\eta^9+\\94564329637587712\eta^{10}+252898202773572096\eta^{11}\\+531724549569874944\eta^{12}+860394900325144576\eta^{13}\\+1034585715180920832\eta^{14}+870984049321230336\eta^{15}\\+458273652727709696\eta^{16}+113442967742775296\eta^{17}\end{pmatrix}}{A},$$

$$P_3=\frac{\begin{pmatrix}9644670+737029935\eta+26623510110\eta^2+603896679360\eta^3\\+9637374939120\eta^4+114894582874272\eta^5+1060377842044928\eta^6\\+7746003075839936\eta^7+45400650083687296\eta^8+215095712322151680\eta^9\\+825648501617611264\eta^{10}+2561494350748951552\eta^{11}\\+6374610597775321088\eta^{12}+12551652909577216000\eta^{13}\\+19121520822631481344\eta^{14}+21744374444976930816\eta^{15}\\+17378376162529181696\eta^{16}+8709235602019319808\eta^{17}\\+2059423493025431552\eta^{18}\end{pmatrix}}{2A},$$

$$P_4=\frac{\begin{pmatrix}4822335+386770950\eta+14732651142\eta^2+354148224300\eta^3\\+6020943361776\eta^4+76896547944768\eta^5+764848862200576\eta^6\\+6061286527831424\eta^7+38827857245143296\eta^8+202779886300886272\eta^9\\+866776699429825536\eta^{10}+3031811716195476480\eta^{11}\\+8640392801203513344\eta^{12}+19884009478335844352\eta^{13}\\+36404297947524317184\eta^{14}+51806358816961101824\eta^{15}\\+55248106337919959040\eta^{16}+41547357299083837440\eta^{17}\\+19648585272683921408\eta^{18}+4395450198923411456\eta^{19}\end{pmatrix}}{A},$$

$$P_5=\frac{\begin{pmatrix}5786802+478395315\eta+18854752482\eta^2+470914789392\eta^3\\+8356099689360\eta^4+111930651620928\eta^5+1173852666958080\eta^6\\+9864943386814528\eta^7+67440297659306624\eta^8+378568916494974720\eta^9\\+1753628099404052992\eta^{10}+6712125721392071680\eta^{11}\\+21181605789714331648\eta^{12}+54786825764960911360\eta^{13}\\+114964449609935060992\eta^{14}+192620700025258049536\eta^{15}\\+251541344253268131840\eta^{16}+246641906422526246912\eta^{17}\\+170750126610003263488\eta^{18}+74384831625259646976\eta^{19}\\+15328376449029636096\eta^{20}\end{pmatrix}}{2A},$$

$$P_6=\frac{\begin{pmatrix}964467+80838081\eta+3238669251\eta^2+82470227922\eta^3\\+1497010364964\eta^4+20590224159960\eta^5+222642500478320\eta^6\\+1937906423421472\eta^7+13789365046902464\eta^8+81004436572247424\eta^9\\+395052384548275200\eta^{10}+1602852501286129664\eta^{11}\\+5404531973044583424\eta^{12}+15079736634245756928\eta^{13}\\+34546415190453469184\eta^{14}+64194781527886708736\eta^{15}\\+95031831980006162432\eta^{16}+109142799900373057536\eta^{17}\\+93408887537593483264\eta^{18}+55801435393530789888\eta^{19}\\+20622984424878243840\eta^{20}+3514385150132617216\eta^{21}\end{pmatrix}}{A},$$

$$P_7=\frac{\begin{pmatrix}275562+23048793\eta+922612194\eta^2+23507343720\eta^3\\+427711600368\eta^4+5909128384752\eta^5+64339900368352\eta^6\\+565512119070400\eta^7+4076204157164160\eta^8+24339343255939840\eta^9\\+121096428605260288\eta^{10}+503183373567916032\eta^{11}\\+1744689526345007104\eta^{12}+5027600355175223296\eta^{13}\\+11950937819425972224\eta^{14}+23161421974674882560\eta^{15}\\+35970414631180271616\eta^{16}+43640591992815747072\eta^{17}\\+39793720202662510592\eta^{18}+25609553049671958528\eta^{19}\\+10350908157396516864\eta^{20}+1971601553789812736\eta^{21}\end{pmatrix}}{2A},\tag{D-2}$$

$$Q_1=\frac{\begin{pmatrix}1653372+109417797\eta+3382785990\eta^2+64937529576\eta^3\\+867199181616\eta^4+8549378203968\eta^5+64402967304384\eta^6\\+378318215194560\eta^7+1751920243563648\eta^8+6418085187943168\eta^9\\+18540636759362048\eta^{10}+41790774444260352\eta^{11}\\+72045568990689280\eta^{12}+91820528698499072\eta^{13}\\+81572044560236544\eta^{14}+45121005328334848\eta^{15}\\+11703985675108352\eta^{16}\end{pmatrix}}{2A},$$

$$Q_2=\frac{\begin{pmatrix}4133430+303807105\eta+10489726800\eta^2+225925740108\eta^3\\+3399803933184\eta^4+37942942782240\eta^5+325289584416192\eta^6\\+2188793723600064\eta^7+11705786899064576\eta^8+50052690610394624\eta^9\\+171141833374995456\eta^{10}+465325419630354432\eta^{11}\\+993795038322507776\eta^{12}+1632157693880070144\eta^{13}\\+1990550846882070528\eta^{14}+1698542582942793728\eta^{15}\\+905298772027637760\eta^{16}+226885935485550592\eta^{17}\end{pmatrix}}{A},$$

$$Q_3=\frac{\begin{pmatrix}2755620+217989225\eta+8158909680\eta^2+191851861176\eta^3\\+3174159248496\eta^4+39219283403424\eta^5+374907497110720\eta^6\\+2834295024300480\eta^7+17175709218640384\eta^8+84046345098126848\eta^9\\+332854530831376896\eta^{10}+1064303775477379072\eta^{11}\\+2727043434501531648\eta^{12}+5523024441354780672\eta^{13}\\+8646264520248131584\eta^{14}+10094785847521574912\eta^{15}\\+8276345505569439744\eta^{16}+4251512462200012800\eta^{17}\\+1029711746512715776\eta^{18}\end{pmatrix}}{2A},$$

$$Q_4=\frac{\begin{pmatrix}2066715+171537345\eta+6778365930\eta^2+169434984816\eta^3\\+3002077365120\eta^4+40036242451824\eta^5+416498844425664\eta^6\\+3456502232840064\eta^7+23207798452598400\eta^8+127108733855598848\eta^9\\+569950215948410368\eta^{10}+2091370860428384256\eta^{11}\\+6251793335977125888\eta^{12}+15087030637713272832\eta^{13}\\+28955259496989720576\eta^{14}+43176987003163705344\eta^{15}\\+48226279773848666112\eta^{16}+37966439939590324224\eta^{17}\\+18787377306036600832\eta^{18}\\+4395450198923411456\eta^{19}\end{pmatrix}}{A},$$

$$Q_5=\frac{\begin{pmatrix}1653372+140831865\eta+5736479130\eta^2+148573325952\eta^3\\+2743890242256\eta^4+38403622770624\eta^5+422530628965440\eta^6\\+3740588253740352\eta^7+27046703004752768\eta^8+161200768478765312\eta^9\\+795729291170984448\eta^{10}+3256554342593625088\eta^{11}\\+11022427468353198080\eta^{12}+30666109237382897664\eta^{13}\\+69400981301546450944\eta^{14}+125722489230917206016\eta^{15}\\+177939942859011391488\eta^{16}+189545896896302809088\eta^{17}\\+142899521909823897600\eta^{18}+67959595475302612992\eta^{19}\\+15328376449029636096\eta^{20}\end{pmatrix}}{2A},$$

$$Q_6=\frac{\begin{pmatrix}275562+23599917\eta+968709780\eta^2+25352568108\eta^3\\+474726287808\eta^4+6764551585488\eta^5+76158157137856\eta^6\\+694180899846528\eta^7+5206626378940160\eta^8+24763549167078 40\eta^9\\+169530574381744128\eta^{10}+742678795901539328\eta^{11}\\+2729167933561139200\eta^{12}+8382009375843201024\eta^{13}\\+21362176345632604160\eta^{14}+44666685497400590336\eta^{15}\\+75336681852947202048\eta^{16}+99963235232629850112\eta^{17}\\+100445529003762384896\eta^{18}+71824989349849595904\eta^{19}\\+32556119303028473856\eta^{20}+7028770300265234432\eta^{21}\end{pmatrix}}{A}.\tag{D-3}$$

REFERENCES

Aleixo, R., and J. Schleicher, 2010, Traveltime approximations for q-P waves in vertical transversely isotropic media: Geophysical Prospecting, **58**, 191–201, doi: 10.1111/j.1365-2478.2009.00815.x.

Alkhalifah, T., 1997, Velocity analysis using nonhyperbolic moveout in transversely isotropic media: Geophysics, **62**, 1839–1854, doi: 10.1190/1.1444285.

Alkhalifah, T., 1998, Acoustic approximations for processing in transversely isotropic media: Geophysics, **63**, 623–631, doi: 10.1190/1.1444361.

Alkhalifah, T., 2000, An acoustic wave equation for anisotropic media: Geophysics, **65**, 1239–1250, doi: 10.1190/1.1444815.

Alkhalifah, T., and I. Tsvankin, 1995, Velocity analysis for transversely isotropic media: Geophysics, **60**, 1550–1566, doi: 10.1190/1.1443888.

Aptekarev, A. I., V. I. Buslaev, A. Martínez-Finkelshtein, and S. P. Suetin, 2011, Padé approximants, continued fractions, and orthogonal polynomials: Russian Mathematical Surveys, **66**, 1049–1131, doi: 10.1070/RM2011v066n06ABEH004770.

Baker, G. A., Jr., 1975, Essentials of Padé approximants in theoretical physics: Academic Press.

Blias, E., 2009, Long-offset NMO approximations for a layered VTI model. Model study: 79th Annual International Meeting, SEG, Expanded Abstracts, 3745–3749.

Castle, R. J., 1994, Theory of normal moveout: Geophysics, **59**, 983–999, doi: 10.1190/1.1443658.

de Bazelaire, E., 1988, Normal moveout revisited-inhomogeneous media and curved interfaces: Geophysics, **53**, 143–157, doi: 10.1190/1.1442449.

Dewangan, P., and I. Tsvankin, 2006, Velocity-independent layer stripping of PP and PS reflection traveltimes: Geophysics, **71**, no. 4, U59–U65, doi: 10.1190/1.2210975.

Dix, C. H., 1955, Seismic velocities from surface measurements: Geophysics, **20**, 68–86, doi: 10.1190/1.1438126.

Douma, H., and A. Calvert, 2006, Nonhyperbolic moveout analysis in VTI media using rational interpolation: Geophysics, **71**, no. 3, D59–D71, doi: 10.1190/1.2194901.

Douma, H., and M. van der Baan, 2008, Rational interpolation of qP-traveltimes for semblance-based anisotropy estimation in layered VTI media: Geophysics, **73**, no. 4, D53–D62, doi: 10.1190/1.2938036.

Fomel, S., 2004, On anelliptic approximations for qP velocities in VTI media: Geophysical Prospecting, **52**, 247–259, doi: 10.1111/j.1365-2478.2004.00413.x.

Fomel, S., and A. Stovas, 2010, Generalized nonhyperbolic moveout approximation: Geophysics, **75**, no. 2, U9–U18, doi: 10.1190/1.3334323.

Fowler, P. J., 2003, Practical VTI approximations: A systematic anatomy: Journal of Applied Geophysics, **54**, 347–367, doi: 10.1016/j.jappgeo.2002.12.002.

Golikov, P., and A. Stovas, 2012, Accuracy comparison of nonhyperbolic moveout approximations for qP-waves in VTI media: Journal of Geophysics and Engineering, **9**, 428–432, doi: 10.1088/1742-2132/9/4/428.

Hake, H., K. Helbig, and C. S. Mesdag, 1984, Three-term Taylor series for t^2-x^2-curves of P- and S-waves over layered transversely isotropic ground: Geophysical Prospecting, **32**, 828–850, doi: 10.1111/j.1365-2478.1984.tb00742.x.

Hubral, P., and T. Krey, 1980, Interval velocities from seismic reflection time measurements: SEG.

Jones, L. E. A., and H. F. Wang, 1981, Ultrasonic velocities in Cretaceous shales from the Williston Basin: Geophysics, **46**, 288–297, doi: 10.1190/1.1441199.

Ma, Z. H., 2005, Modern applied mathematics handbook (in Chinese): Tsinghua University Press.

Malovichko, A. A., 1978, A new representation of the traveltime curve of reflected waves in horizontally layered media: Applied Geophysics (in Russian), **91**, 47–53.

Ravve, I., and Z. Koren, 2009, Moveout approximation for HTI/VTI layered media: 79th Annual International Meeting, SEG, Expanded Abstracts, 281–284, doi: 10.1190/1.3255435.

Ravve, I., and Z. Koren, 2010, Moveout approximation for horizontal transversely isotropic and vertical transversely isotropic layered medium. Part I: 1D ray propagation: Geophysical Prospecting, **58**, 577–597, doi: 10.1111/j.1365-2478.2009.00856.x.

Ravve, I., and Z. Koren, 2015, Long offset asymptotic moveout for compressional waves in layered orthorhombic media: 77th Annual International Conference and Exhibition, EAGE, Extended Abstracts, 2246–2250, doi: 10.3997/2214-4609.201413045.

Siliqi, R., 2001, Technological leap in time processing focuses the data throughout anisotropic media: First Break, **19**, 612–618.

Siliqi, R., and N. Bousquie, 2000, Anelliptic time processing based on a shifted hyperbola approach: 70th Annual International Meeting, SEG, Expanded Abstracts, 2245–2248, doi: 10.1190/1.1815902.

Stovas, A., 2010, Generalized moveout approximation for qP- and qSV-waves in a homogeneous transversely isotropic medium: Geophysics, **75**, no. 6, D79–D84, doi: 10.1190/1.3506504.

Stovas, A., and B. Ursin, 2004, New travel-time approximations for a transversely isotropic medium: Journal of Geophysics and Engineering, **1**, 128, doi: 10.1088/1742-2132/1/2/004.

Taner, M. T., and F. Koehler, 1969, Velocity spectra-digital computer derivation and applications of velocity functions: Geophysics, **34**, 859–881, doi: 10.1190/1.1440058.

Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954–1966, doi: 10.1190/1.1442051.

Tsvankin, I., and L. Thomsen, 1994, Nonhyperbolic reflection moveout in anisotropic media: Geophysics, **59**, 1290–1304, doi: 10.1190/1.1443686.

Tsvankin, I., and L. Thomsen, 1995, Inversion of reflection traveltimes for transverse isotropy: Geophysics, **60**, 1095–1107, doi: 10.1190/1.1443838.

Tsvankin, I., 2001, Seismic signatures and analysis of reflection data in anisotropic media: Elsevier Science Publishing Company Inc.

Ursin, B., and A. Stovas, 2006, Traveltime approximations for a layered transversely isotropic medium: Geophysics, **71**, no. 2, D23–D33, doi: 10.1190/1.2187716.

Wang, X., and I. Tsvankin, 2009, Estimation of interval anisotropy parameters using velocity independent layer stripping: Geophysics, **74**, no. 5, WB117–WB127, doi: 10.1190/1.3157462.